ON UNITARY DEFORMATIONS OF SMOOTH MODULAR REPRESENTATIONS

TOBIAS SCHMIDT

Abstract. Let G be a locally $\mathbb{Q}_p$-analytic group and K a finite extension of $\mathbb{Q}_p$ with residue field k. Generalizing results of B. Mazur we use deformation theory to study the possible liftings of a given smooth G-representation over k to unitary G-Banach space representations over K. We compute some explicit deformations in the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

1. Introduction

Let G a locally $\mathbb{Q}_p$-analytic group and K a finite extension of $\mathbb{Q}_p$ with ring of integers o and residue field k. The aim of the present note is to study the set of possible liftings of a given smooth G-representation ρ over k to unitary G-Banach space representations over K. To do this we generalize the techniques of deformation theory for representations of profinite groups as developed by B. Mazur (cf. [Maz89]) to our present situation. We prove, in case ρ is admissible and absolutely irreducible, the existence of a formal scheme Spf $R(G, K, \rho)$ over o which depends functorially on the datum (G, K, ρ) and respects elementary operations on ρ such as tensor product or contragredient. Moreover, its o-rational points biject canonically with the isomorphism classes of unitary liftings of ρ. The ring $R(G, K, \rho)$ is a local profinite o-algebra with residue field k which is noetherian if and only if the k-vector space of extensions $\mathrm{Ext}^1_G(\rho, \rho)$ is of finite dimension. Basic features of $R(G, K, \rho)$ such as its dimension or its irreducible components remain unclear at this point.

The ring $R(G, K, \rho)$ represents a deformation problem for Iwasawa modules which is based on the simple observation (cf. [Pas]) that the duality functor of P. Schneider and J. Teitelbaum (cf. [ST02]) on the category of unitary representations is compatible with Pontryagin duality. By work of M. Emerton (cf. [Emea]) the dual categories admit generalizations to complete local noetherian o-algebras which provide a natural framework to study deformations of (the Pontryagin dual of) ρ.

Let us briefly outline the paper. For conceptual clarity we proceed axiomatically and fix an arbitrary complete local noetherian ring o with residue field k of characteristic $p > 0$ (finite or not) and a smooth G-representation ρ over k. We introduce a category of coefficient rings consisting of local pseudocompact o-algebras A such that $o \to A$ gives an isomorphism on residue fields. Results of A. Brumer (cf. [Bru66]) on completed group algebras over commutative pseudocompact rings allow to introduce categories of smooth G-representations over such A and an appropriate duality (generalizing Emerton's constructions). The presence of a well-defined base change functor on these categories results in a deformation functor D_ρ. We prove its representability if ρ is admissible and absolutely irreducible and give the usual Ext^1_G-criterion for $R(G, K, \rho)$ to be noetherian.

We remark straightaway that M. Schlessinger's method (cf. [Sch68]) for proving representability is not applicable due to our reluctance from any finiteness assumption on the tangent space of D_ρ. Instead, we proceed directly from A.

Grothendieck's theorem (cf. [Gro]) thereby generalizing ideas of M. Dickinson (cf. [Gou01]).

We proceed by computing the universal deformation in case of a smooth character. We explain how our theory reduces to the situation considered by B. Mazur when G is compact and k is finite. We explain the meaning of deformation conditions in our setting. We then prove the usual functorial properties of the universal deformation ring with respect to change of group, field, range, twisting and tensor product.

In the final section we turn to Banach space representations and specialize the deformation theory to the situation where o is given as ring of integers in K. We finish this work with an application to the group $G = GL_2(\mathbb{Q}_p)$ and compute the set of unitary deformations in case ρ equals a principal series representation.

Acknowledgements. The author would like to thank the Arbeitsgruppe of Peter Schneider at Münster University whose Oberseminar in the winter term 2005/06 provided a motivating opportunity for the author to learn about deformation theory and its applications to group representations.

2. The deformation problem

For any unital ring A we let $\mathfrak{M}(A)$ be the abelian category of left unital o-modules. If A is left noetherian then the finitely generated left A-modules form a full subcategory $\mathfrak{M}_{fg}(A)$ of $\mathfrak{M}(A)$. For basic notions on pseudocompact rings we refer to the appendix. The symbol G always denotes a locally $\mathbb{Q}_p$-analytic group.

2.1. **Completed group algebras.** Let A be a commutative pseudocompact ring and $H \subseteq G$ a fixed compact open subgroup. A topological A-module H is called a *left H-module* (over A) if it has a H-action by continuous A-linear maps such that the map

$$H \times M \longrightarrow M$$

giving the action is continuous. Writing $\mathcal{N}$ for the system of open normal subgroups of H we denote by

$$A[[H]] := \varprojlim_{N \in \mathcal{N}} A[H/N]$$

the completed group algebra of H over A. It is a pseudocompact A-algebra with respect to the projective limit topology and the correspondance $H \mapsto A[[H]]$ is a covariant functor from profinite groups to pseudocompact A-algebras (cf. [Bru66], Sect. 2). The anti-involution $h \mapsto h^{-1}$ on H identifies $A[[H]]$ and $A[[H]]^{opp}$ as pseudocompact A-algebras making it unnecessary to distinguish between left and right modules. Let $\mathfrak{PM}(A[[H]])$ be the abelian category of (left) pseudocompact $A[[H]]$-modules (cf. appendix).

Lemma 2.1. *If A is noetherian then $A[[H]]$ is noetherian.*

Proof: If A is noetherian its pseudocompact topology coincides with the adic topology defined by the Jacobson radical of A. Arguing as in [Emea], Prop. 2.1.2 we may therefore deduce that $A[[H]]$ is noetherian. $\square$

A discrete A-module of finite length M is an H-module if and only if it is a (discrete) $A[[H]]$-module (cf. [Bru66], Sect. 4). A projective limit argument shows that a pseudocompact A-module M is an H-module if and only if it is a (pseudocompact) $A[[H]]$-module. Given $M, N \in \mathfrak{PM}(A[[H]])$ the diagonal H-action on the pseudocompact A-module $M \hat{\otimes}_A N$ is evidently continuous and therefore extends to a pseudocompact $A[[H]]$-module structure. The resulting binary operation $\hat{\otimes}_A$

on $\mathfrak{PM}(A[[H]])$ is associative, commutative and functorial in both variables. The usual augmentation homomorphism $A[[H]] \to A$ provides a unit object.

Consider a pseudocompact A-algebra B and let $\phi : A \to B$ be the structure map. The base change $\phi^* : \mathfrak{PM}(A) \to \mathfrak{PM}(B)$ commutes with projective limits. Thus, the compatible system of natural isomorphisms

$$B \otimes_A A[H/N] \xrightarrow{\cong} B[H/N],\ N \in \mathcal{N}$$

induces a natural isomorphism $\phi^*(A[[H]]) \xrightarrow{\cong} B[[H]]$ which is multiplicative (note that the left-hand side equals the amalgamated sum of the structure maps $A \to A[[H]]$ and ϕ along A). This discussion yields a functor

$$\phi_H^* : \mathfrak{PM}(A[[H]]) \to \mathfrak{PM}(B[[H]])$$

compatible with ϕ^* via the forgetful functors (17). Given another pseudocompact B-algebra B' with structure map ψ we evidently have

$$(\psi \circ \phi)_H^* = \psi_H^* \circ \phi_H^*. \tag{1}$$

Let additionally, $\mathfrak{PM}(A[[H]])^{\mathrm{fl}}$ denote the full subcategory of $\mathfrak{PM}(A[[H]])$ consisting of modules on underlying topologically free A-modules (similarly for B). The functor ϕ_H^* respects these subcategories.

2.2. **Augmentations.** We now introduce certain categories of G-representations. These are adapted versions of categories first introduced by M. Emerton in [Emea]. In view of applications we will work over a fixed commutative complete local noetherian ring o.

Let A be a commutative pseudocompact o-algebra and denote by $A[G]$ the usual group algebra of G over A. A *G-representation over A* is simply a (left) $A[G]$-module. Such a representation M is called *augmented* if the induced $A[H]$-action extends to an $A[[H]]$-action on M for every compact open subgroup H of G. With morphisms being G-equivariant maps that are simultaneously $A[[H]]$-linear for every compact open subgroup H of G such representations form an abelian category. An augmented G-representation M is called a *pseudocompact augmented* G-representation over A if the underlying $A[H]$-module structure extends to an $A[[H]]$-module structure yielding an object in $\mathfrak{PM}(A[[H]])$ for every compact open subgroup H of G.

Lemma 2.2. *The pseudocompact topology on M is independent of the choice of H.*

Proof: The ring $A[[H]]$ is a pseudocompact A-algebra and we have a well-defined forgetful functor $\mathfrak{PM}(A[[H]]) \to \mathfrak{PM}(A)$ according to (17). $\square$

We define a morphism between two such representations to be a G-equivariant A-linear map and denote the resulting category by

$$\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A).$$

Proposition 2.3. *The category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is an abelian tensor category.*

Proof: Since $A[H] \subseteq A[[H]]$ is dense any morphism induces a morphism in $\mathfrak{PM}(A[[H]])$ for every compact open subgroup H of G. Now $\mathfrak{PM}(A[[H]])$ is abelian and kernels and cokernels are preserved by the forgetful functor to $\mathfrak{PM}(A)$. It follows easily that $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is abelian. Finally, given $M, N \in \mathfrak{PM}(A[[H]])$ the diagonal G-action on

$$M \hat{\otimes}_A N \in \mathfrak{PM}(A[[H]])$$

for any compact open subgroup $H \subseteq G$ makes the latter module evidently a pseudocompact augmented G-representation over A. This yields the desired tensor product on $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$. $\square$

Remark: If A is noetherian with finite residue field and o is a compact complete discrete valuation ring we recover the definition of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ in [Emea].

Let E denote the *dualiser* of o, i.e. the injective envelope, in the category of discrete o-modules, of the residue field of o (cf. [Bru66], §2). Given $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ the *dual*

$$M^\vee := \mathrm{Hom}_o^{cont}(M, E)$$

is a discrete A-module endowed with the contragredient G-action.

Lemma 2.4. *Passing to duals*

$$M \mapsto M^\vee$$

induces an equivalence between $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ *and the category* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ *of discrete G-modules over A.*

Proof: Passage to duals induces an equivalence between $\mathfrak{PM}(A[[H]])$ and discrete $A[[H]]$-modules for each compact open subgroup $H \subseteq G$ (cf. [Bru66], Prop.2.3). The claim is an easy consequence of the fact that these equivalences respect the forgetful functors to $\mathfrak{PM}(A)$ and to discrete A-modules respectively. $\square$

Remark: If o is a compact complete discrete valuation ring the dualiser of o equals K/o where K is the fraction field of o. The above duality is therefore induced by usual Pontryagin duality over the profinite ring o.

Suppose A is noetherian. Let us write $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ for the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ consisting of objects that are finitely generated over $A[[H]]$ for any compact open subgroup H of G. By uniqueness of the pseudocompact topology on finitely generated modules (cf. appendix) we obtain a natural fully faithful and exact embedding

$$\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \xrightarrow{\subseteq} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A).$$

We denote the image of this category under the above duality by $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and refer to its objects as *admissible* discrete G-modules. We thus have an equivalence

$$(.)^\vee : \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \xrightarrow{\cong} \mathrm{Mod}_G^{\mathrm{adm}}(A). \tag{2}$$

Since A is noetherian its pseudocompact topology equals the $\mathfrak{m}$-adic topology where $\mathfrak{m}$ denotes the Jacobson radical of A. In this case (cf. [Emea]) we have the following simple descriptions of the categories $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ and $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Given a G-representation V a vector $v \in V$ is called *smooth* if it is fixed by an open subgroup of G and annihilated by some power $\mathfrak{m}^i$ of the maximal ideal $\mathfrak{m}$ of A. The representation V is called *smooth* if all vectors $v \in V$ are smooth. We see that a G-representation V is smooth if and only if $V \in \mathrm{Mod}_G^{\mathrm{sm}}(A)$. Keeping our assumptions on A a smooth G-representation V is called *admissible* if the $\mathfrak{m}^i$-torsion part of the subspace of H-fixed vectors in V is finitely generated over A for every $i \geq 0$ and every open subgroup H of G. We see that a smooth G-representation V is admissible if and only if $V \in \mathrm{Mod}_G^{\mathrm{adm}}(A)$.

Remark: If A is artinian (e.g. a field) then $\mathfrak{m}^i = 0$ for some $i \geq 0$ and the definitions of *smooth* and *admissible* reduce to the usual ones in representation theory of locally profinite groups. More specifically, if A equals the residue field k of o and $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ then $M^\vee \simeq \mathrm{Hom}_k^{cont}(M, k) = M^*$ canonically (cf. (14)).

2.3. **Functors on coefficient algebras.** Let as before o be a commutative complete local noetherian ring. We now define two subcategories of pseudocompact o-algebras which will serve as coefficient algebras within the upcoming deformation theory.

Let $\mathfrak{m}$ be the maximal ideal of o with residue field k. Let $\hat{C}$ be the full subcategory of local pseudocompact o-algebras A such that the structure map $o \to A$ induces an isomorphism on residue fields. In particular, $\mathfrak{m}A$ equals the maximal ideal of A which is therefore finitely generated. Let C denote the full subcategory of $\hat{C}$ consisting of discrete algebras having finite length as o-module. Without recalling the precise definition of a *pro-object* (cf. [Gro], A.2) we have the following lemma.

Lemma 2.5. *The category of pro-objects of C is equivalent to $\hat{C}$.*

Proof: Let us denote by $Pro(.)$ the passage from a category to its pro-objects. Let C' be the category of all discrete o-algebras having finite length as o-module. Mapping a pseudocompact o-algebra to the system of all its artinian quotients induces an equivalence between pseudocompact o-algebras and $Pro(C')$ (cf. [Gro], A.5). Restricting this functor to $\hat{C}$ yields a fully faithful functor into $Pro(C)$. Conversely, given an element $(R_i)_i$ in $Pro(C)$ the projective limit $\varprojlim_i R_i$ is a pseudocompact o-algebra which is easily seen to be local with prescribed residue isomorphism. This gives a quasi-inverse. $\square$

We now bring in a set-valued covariant functor on C

$$D : C \longrightarrow Sets.$$

The category C contains k as a terminal object and admits finite products and finite fiber products (cf. [Maz97], Lem. IV.§14). As to the latter, recall that if $\phi_i : A_i \to A_0$ are two morphisms in C their fiber product is given as the equalizer

$$A_1 \times_{A_0} A_2 = \{(a_1, a_2) \in A_1 \times A_2 : \phi_1(a_1) = \phi_2(a_2)\}$$

with ring structure induced from $A_1 \times A_2$. In this situation D is called *left exact* if it respects finite products and finite fiber products. Furthermore, since $\hat{C}$ identifies with the pro-objects of C, the functor D being *pro-representable* is tantamount to being of the form $\mathrm{Hom}_{\hat{C}}(R, .)$ with some $R \in \hat{C}$ (cf. [Gro], A.2).

Let $k[\epsilon] = k[x]/x^2$ be the ring of dual numbers viewed as an object in C. If D is pro-representable the set $D(k[\epsilon])$ evidently has a natural k-vector space structure (the "tangent space" of D).

Theorem 2.6. *The functor $D : C \to Sets$ is pro-representable if and only if it is left exact. In this situation the representing ring R is noetherian if and only if the k-vector space $D(k[\epsilon])$ has finite dimension d. In this case, R equals a quotient of the formal power series ring $o[[x_1, ..., x_d]]$.*

Proof: This follows directly from A. Grothendieck's representability theorem (cf. [Gro], Prop. A.3.1/A.5.1). $\square$

Suppose we now have a functor $D : \hat{C} \to Sets$ on the larger category $\hat{C}$. By the above lemma $\hat{C}$ is stable under arbitrary projective limits. The above discussion therefore shows that D is representable as a functor on $\hat{C}$ if and only if it commutes with projective limits and the restriction of D to C is pro-representable.

2.4. **Deformations.** We define the deformation problem and state the main representability result. We keep the assumptions of the previous subsection but **assume** that the residue field of o has characteristic $p > 0$.

To start with consider the full subcategory $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ consisting of representations on topologically free underlying A-modules. Forgetting the group actions for a moment we record a stability property of topologically free modules for which we have not found an appropriate reference.

Lemma 2.7. *Let $A = \varprojlim_n A_n \in \hat{C}$ with $A_n \in C$ an artinian pseudocompact quotient of A for all n. Let $(M_n)_n$ be a projective system where each M_n is a pseudocompact topologically free A_n-module. The transition map $M_{n+1} \to M_n$ is supposed to be continuous and compatible with $A_{n+1} \to A_n$. Then $M := \varprojlim_n M_n$ equipped with the projective limit topology is a pseudocompact topologically free A-module.*

Proof: Via the quotient map $A \to A_n$ we may view each M_n as a pseudocompact A-module. It follows that M is a pseudocompact A-module and, according to the proof of [DG70], Prop. 0.3.7, that M is topologically free over A if the autofunctor $(.)\,\hat{\otimes}_A M$ on $\mathfrak{PM}(A)$ is exact. So suppose that

$$\mathcal{E} : 0 \longrightarrow N' \longrightarrow M' \xrightarrow{\phi} P' \longrightarrow 0$$

is a short exact sequence in $\mathfrak{PM}(A)$. Let $\mathfrak{m}_n$ be the kernel of $A \to A_n$ and put $M_n' = \overline{\mathfrak{m}_n M'}$. Since $\mathfrak{m}_n$ is closed it follows easily from [DG70], 0.3.2 that $M' = \varprojlim_n M'/M_n'$. Putting $N_n' = N \cap M_n'$ and $P_n' = \phi(M_n')$ yields the exact sequence

$$\mathcal{E}_n : 0 \longrightarrow N'/N_n' \longrightarrow M'/M_n' \xrightarrow{\phi} P'/P_n' \longrightarrow 0$$

of artinian A-modules for all n. Since $\cap M_n' = 0$ we have $\cap N_n' = \cap P_n' = 0$ and $\varprojlim_n \mathcal{E}_n = \mathcal{E}$ by exactness of $\varprojlim_n$ on $\mathfrak{PM}(A)$. Since $\hat{\otimes}_A$ commutes with projective limits we obtain isomorphisms of topological A-modules

$$\mathcal{E}\,\hat{\otimes}_A M \xrightarrow{\cong} \varprojlim_n \mathcal{E}_n \,\hat{\otimes}_A M_n \xrightarrow{\cong} \varprojlim_n \mathcal{E}_n \,\hat{\otimes}_{A_n} M_n.$$

Since M_n is topologically free over A_n for all n the functor $(.)\,\hat{\otimes}_A M$ is seen to be exact. $\square$

Given a pseudocompact A-algebra B let $\phi : A \to B$ be the structure map. For every $H \in \mathcal{N}$ we have the base change ϕ_H^* commuting with forgetful functors and, hence, a functor

$$\phi_G^* : \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \longrightarrow \mathrm{Mod}_G^{\mathrm{pro\,aug}}(B) \tag{3}$$

respecting the subcategories $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ and $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(B)^{\mathrm{fl}}$. Given another pseudocompact B-algebra B' with structure map ψ one has

$$(\psi \circ \phi)_G^* = \psi_G^* \circ \phi_G^* \tag{4}$$

according to (1).

After these preliminaries let us fix once and for all a smooth G-representation ρ over k with dual

$$N = \rho^\vee \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k).$$

Let I be an index set of a pseudobasis for the topologically free k-module underlying N. Invoking the duality $N \mapsto N^*$ (cf. (14)) we see that the cardinality $|I|$ does not depend on the choice of pseudobasis.

Given a local pseudocompact o-algebra $A \in \hat{C}$ with residue homomorphism $\phi : A \to k$ we consider couples (M, α) such that $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ and

$$\alpha : k\,\hat{\otimes}_A M = \phi_G^*(M) \xrightarrow{\cong} N$$

is an isomorphism in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)^{\mathrm{fl}}$.

Lemma 2.8. *Given $M \in \mathfrak{PM}(A)$ the subset $\mathfrak{m}M \subseteq M$ is closed and the natural map*

$$k \hat{\otimes}_A M \longrightarrow M/\mathfrak{m}M$$

is an isomorphism in $\mathfrak{PM}(k)$.

Proof: The A-submodule $\mathfrak{m}M \subseteq M$ equals the image of a morphism in $\mathfrak{PM}(A)$ of the form $\prod M \to M$ where the product is indexed by finitely many generators of the ideal $\mathfrak{m}$. It is therefore closed. The map $A \to k$ gives rise to an exact sequence in $\mathfrak{PM}(A)$

$$\mathfrak{m} \hat{\otimes}_A M \longrightarrow M \longrightarrow k \hat{\otimes}_A M \longrightarrow 0.$$

The image of $\mathfrak{m} \hat{\otimes}_A M$ in M is closed and contains $\mathfrak{m}M$ as a dense subset (cf. [DG70], 0.3.2). □

A morphism of couples $(M, \alpha) \to (M', \alpha')$ is a morphism $M \to M'$ in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ such that the resulting diagram

$$\begin{array}{ccc} M/\mathfrak{m}M & \xrightarrow[\alpha]{\sim} & N \\ \downarrow & & \downarrow = \\ M'/\mathfrak{m}M' & \xrightarrow[\alpha']{\sim} & N \end{array} \tag{5}$$

is commutative. We denote the set of isomorphism classes of such couples by $D_\rho(A)$. Elements in $D_\rho(A)$ will be called *deformations* of N to A and we will often abbreviate $M = [M, \alpha]$ when no confusion can arise. By associativity (4) of the base change ϕ_G^* we obtain a contravariant set-valued functor

$$D_\rho : \hat{C} \longrightarrow Sets$$

such that $D_\rho(k)$ is a singleton. Let $\mathrm{Ext}_G^1(\rho, \rho)$ denote the k-vector space of *Yoneda extensions* of ρ by itself in the abelian category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$.

Proposition 2.9. *There is a natural bijection*

$$D_\rho(k[\epsilon]) \xrightarrow{\simeq} \mathrm{Ext}_G^1(\rho, \rho)$$

which is k-linear in case D_ρ is representable.

Proof: This is a standard phenomenon in deformation theory (cf. [Maz97], Prop. V.§22). Let $A = k[\epsilon]$ with $\pi : A \to k$ and $\iota : k \to A$ the canonical maps. Let $[M, \alpha] \in D_\rho(A)$. Invoking an isomorphism $M \simeq A^I$ in $\mathfrak{PM}(A)$ we have $\epsilon M \simeq \prod_I k \simeq N$ in $\mathfrak{PM}(k)$. Since multiplication by ϵ is a homeomorphism onto its image the restriction map induced by $\epsilon M \subseteq M$ and ι

$$res : \mathrm{End}_A^{cont}(M) \longrightarrow \mathrm{End}_k^{cont}(\epsilon M)$$

is continuous for our usual topologies (cf. (A.1)). The diagram

$$\begin{array}{ccccc} A[[H]] & \xrightarrow{cont} & \mathrm{End}_A^{cont}(M) & \xrightarrow{res} & \mathrm{End}_k^{cont}(\epsilon M) \\ \uparrow \subseteq & & \uparrow & \nearrow & \\ A[H] & \xrightarrow{\subseteq} & A[G] & & \end{array} .$$

is commutative and factores through the projection π so that $\epsilon M \simeq N$ lifts to an isomorphism $\epsilon M \simeq N$ in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. In other words

$$0 \to \epsilon M \to M \to M/\epsilon M \to 0$$

yields an element in $\mathrm{Ext}^1(N,N)$ (the space of Yoneda extensions in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$). The map $D_\rho(A) \to \mathrm{Ext}^1(N,N)$ is k-linear. To construct the inverse let

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} N \longrightarrow 0 \tag{6}$$

be an extension in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. By topological freeness it splits in $\mathfrak{PM}(k)$ and we may assume that M admits a neighbourhood basis of zero consisting of k-vector spaces stable under the action of $\iota\circ\pi$. Setting $\epsilon.m := (\iota\circ\pi)(m)$ makes M a linearly topologized A-module whence $M \in \mathfrak{PM}(A)$. An easy argument shows that the A-module on the dual k-vector space M^* is free. Applying the quasi-inverse to (14) we find M to be topologically free over A. Finally, since ι, π are linear with respect to $A[[H]]$ and $A[G]$ it follows that $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$. This construction gives the inverse. $\square$

We come to the main result of this section which will be proved later.

Theorem 2.10. *If ρ is absolutely irreducible and admissible then D_ρ is representable.*

Remark: Suppose N is finitely generated over $k[[H]]$ for some (equivalently any) compact open $H \subseteq G$. Then any $M \in D_\rho(A)$ is finitely generated over $A[[H]]$. Indeed, lifting finitely many $k[[H]]$-module generators of $M/\mathfrak{m}M = N$ to M we obtain a map $\prod A[[H]] \to M$ whose cokernel Q satisfies $Q/\mathfrak{m}Q = 0$. Since A is local $Q = 0$ by the Nakayama lemma (cf. [DG70], 0.3.3).

Corollary 2.11. *Let D_ρ be representable. The representing o-algebra R is noetherian if and only if $d := \dim_k \mathrm{Ext}^1_G(\rho,\rho) < \infty$. In this situation R equals a quotient of the formal power series ring $o[[x_1, ..., x_d]]$.*

Proof: This follows from prop. 2.11 and thm. 2.6. $\square$

2.5. **The case of a character.** We will compute the universal deformation ring R and the universal deformation in case of a character-at least if k is finite. We will show in subsection 4.3 that, up to canonical isomorphism, R does not depend on the particular choice $\bar{\chi}$ of character (but the universal deformation does). The argument we use is an adaption of the one given in [Gou01], Prop.3.13.

Our starting point is the standard fact that any commutative local pseudocompact ring is *henselian* (cf. [Nag62], Thm. V.30.3). Suppose A is such a ring with maximal ideal $\mathfrak{m}$ and finite residue field $k = A/\mathfrak{m}$. We have a short split exact sequence

$$1 \longrightarrow 1+\mathfrak{m} \longrightarrow A^\times \longrightarrow k^\times \longrightarrow 1.$$

Now assume k is finite of characteristic $p > 0$ so that o is a local pro-p ring. Let $\Gamma = \Gamma(G)$ be the pro-p completion of the Hausdorff abelianization $G/\overline{[G,G]}$ of G and

$$\gamma : G \longrightarrow G/\overline{[G,G]} \longrightarrow \Gamma$$

be the natural map. The completed group algebra $R = o[[\Gamma]]$ is a commutative local pseudocompact o-algebra (cf. [Bru66], Ex. §4.(2)), in other words $R \in \hat{C}$. We now suppose our residue representation $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ is given by a character $\bar{\chi} : G \to k^\times$. Composing $\bar{\chi}$ and γ with the splitting $k^\times \to o^\times$ and the natural map $\Gamma \to R^\times$ respectively we obtain two maps $\chi_0 : G \to o^\times$ and $[.] : G \to R^\times$.

Proposition 2.12. *The homomorphism*

$$\chi^{univ} = \chi_0 \cdot [.] : G \longrightarrow R^\times$$

equals the universal deformation.

Proof: Fix a compact open subgroup $H \subseteq G$. First note that $H \xrightarrow{\bar{\chi}} k^\times \to o^\times$ is a continuous homomorphism between profinite groups whence a continuous map $R[[H]] \to R[[o^\times]]$. The homomorphism $o^\times \to R^\times$ induced by the algebra structure of R is continuous whence a continuous map $R[[o^\times]] \to R$ by the universal property of o[[.]] applied to the profinite ring R. The composite

$$R[[H]] \longrightarrow R[[o^\times]] \longrightarrow R$$

then coincides with χ_0 on the subring $R[H]$. Secondly, the map [.] restricted to G has image in $1 + \mathfrak{m}_R \subseteq R^\times = k^\times \times (1 + \mathfrak{m}_R)$ since Γ is a pro-p group. On the one hand, this yields a continuous map

$$R[[H]] \longrightarrow R[[1 + m_R]] \longrightarrow R$$

that coincides with [.] on the subring $R[H]$. We therefore find that $\chi^{univ} \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(R)^{\mathrm{fl}}$. On the other hand, it shows that χ^{univ} is a deformation of $\bar{\chi}$.

Now suppose $[\chi]$ is a deformation of $\bar{\chi}$ to $A \in \hat{C}$. Since $1 + m_A$ is an abelian pro-p group the continuous map $\chi_0^{-1} \cdot \chi : G \to 1 + m_A$ factores through Γ. We obtain a continuous map

$$f_\chi : R = o[[\Gamma]] \longrightarrow o[[1 + m_A]] \longrightarrow A$$

that specializes χ^{univ} to χ. It follows that χ^{univ} is the universal deformation. $\square$

We give an example in which R is noetherian.

Corollary 2.13. *Let $\mathbb{G}$ denote a connected reductive group over $\mathbb{Q}_p$ and G its $\mathbb{Q}_p$-rational points. Then R is noetherian.*

Proof: The subgroup $[\mathbb{G}, \mathbb{G}]$ is Zariski-closed in $\mathbb{G}$ and $\mathbb{G}/[\mathbb{G}, \mathbb{G}]$ is a quotient of the connected center of $\mathbb{G}$ (cf. [Bor91], Prop. 14.2). We deduce that $[G, G]$ is closed and $G/[G, G]$ is topologically finitely generated. Hence, so is $\Gamma = \Gamma(G)$ and the universal deformation ring $R = o[[\Gamma]]$, being a quotient of a power series ring in finitely many indeterminates over o, is noetherian. $\square$

Example: If $G = \mathrm{GL}_n(\mathbb{Q}_p)$ the pro-p-completion of

$$G/[G, G] = \mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mu_{p-1} \times U^1$$

equals $\mathbb{Z}_p \times U^1$. The 1-units U^1 of $\mathbb{Z}_p$ are isomorphic to $\mathbb{Z}_p$ as topological group via the logarithm map. It follows that $R = o[[x_1, x_2]]$.

2.6. **The compact case.** Suppose G is compact and k is a *finite* field of characteristic $p > 0$. We show how our deformation theory reduces to the classical theory (cf. [Maz89], [Maz97]).

Given an absolutely irreducible smooth G-representation ρ over k let I be an indexing set for a pseudobase of $N = \rho^\vee$. Since any normal pro-p subgroup H of G acts trivially on the smooth G-representation ρ (cf. [Wilson], Lem. 11.1.1) we have $n = |I| < \infty$ and ρ is evidently admissible. Furthermore, any deformation of N to $A \in \hat{C}$ has a finite free underlying A-module (cf. Lem. 3.1) and thus, D_ρ describes the equivalence classes of continuous lifts

$$G \longrightarrow \mathrm{GL}_n(A)$$

of ρ to A. Since G contains an open pro-p subgroup of finite rank (cf. [DdSMS99], Cor. 8.33) the profinite group G satisfies the p-finiteness condition as formulated in [Maz89]. Thus, a standard argument in Galois cohomology (cf. [Maz97], Prop. §21.2a) yields $d = \dim_k \mathrm{Ext}^1_G(\rho, \rho) < \infty$ and we are exactly in the situation of

[Maz89], Prop.1. It follows that this latter result on the representability of D_ρ is included in our Thm. 3.

We recall from Cor. 2.13 that in this situation the universal deformation ring R equals a quotient of $o[[x_1, ..., x_d]]$. We illustrate a simple case in which actually $R = o[[x_1, ..., x_d]]$. Writing $G_0 := \ker \rho$ let $G_0 \to P$ be the maximal pro-p quotient of G_0 (cf. [RZ00], §3.4). It sits in a short exact sequence

$$0 \to P \to \tilde{G} \to \text{Image } \rho \to 0 \tag{7}$$

where $\tilde{G}$ denotes the so-called p-completion of G with respect to ρ (cf. [Maz89]).

Proposition 2.14. *Assume ρ is tame (i.e. $p \nmid \#\text{Image } \rho$). If P is a free pro-p group one has an isomorphism $o[[x_1, ..., x_d]] \xrightarrow{\cong} R$.*

Proof: Let $\mathrm{Ad}(\rho)$ equal the continuous $\tilde{G}$-module $M_n(k)$ where $\tilde{G} \to \mathrm{GL}_n(k)$ acts by conjugation. Since any lifting of ρ to $A \in \hat{C}$ factores through $\tilde{G}$ the proof of [Gou01], Thm. 4.2 shows that it suffices to show that the continuous cohomology group $H^2(\tilde{G}, \mathrm{Ad}(\rho))$ vanishes. Since $P \subseteq \tilde{G}$ is a pro-p Sylow subgroup and $\mathrm{Ad}(\rho)$ is p-torsion the restriction map $H^2(\tilde{G}, \mathrm{Ad}(\rho)) \to H^2(P, \mathrm{Ad}(\rho))$ is injective (cf. [RZ00], Cor. 6.7.7). But any free pro-p group has cohomological dimension ≤ 1 (cf. [RZ00], Thm. 7.7.4) and so $H^2(P, \mathrm{Ad}(\rho)) = 0$. $\square$

Remark: If ρ is tame the famous Schur-Zassenhaus theorem yields a splitting of (7). This is the starting point of N. Boston's explicit computations of (Galois) deformation rings (cf. [Bos91]).

2.7. **Deformation conditions.** We illustrate that the usual formalism of *deformation conditions* works in our setting. We suppose in the following that the above deformation functor D_ρ is representable.

Given $A \in \hat{C}$ assume that some elements of $D_\rho(A)$ have been designated to be "of type $\mathcal{P}$" and that this property is preserved under the base change $D_\rho(\phi)$ associated to morphisms $\phi : A \to B$ in $\hat{C}$. We obtain a subfunctor

$$\mathcal{D}_\rho \subseteq D_\rho$$

by putting $\mathcal{D}_\rho(A) := \{M \in D_\rho(A) : M \text{ of type } \mathcal{P}\}$ for $A \in \hat{C}$.

Proposition 2.15. *The following conditions are necessary and sufficient for the representability of $\mathcal{D}_\rho$:*

1. *$N \in D_N(k)$ has property $\mathcal{P}$.*
2. *Given a diagram $A_1 \to A_0 \leftarrow A_2$ in C, any deformation of N to the fiber product $A_1 \times_{A_0} A_2$ whose base changes to A_1 and A_2 are of type $\mathcal{P}$ is of type $\mathcal{P}$.*
3. *If $A \in \hat{C}$ is an inverse limit of objects A_i in C and the basechange to A_i of a deformation M of N to A is of type $\mathcal{P}$ for each i then M is of type $\mathcal{P}$.*

Proof: Granting the representability of D_N the first two conditions are tantamount to the fact that $\mathcal{D}_\rho$ is left-exact and the third condition asserts that $\mathcal{D}_\rho$ preserves arbitrary inverse limits. The claim is therefore a direct consequence of Thm. 2.6. $\square$

Let us give two examples of prominent deformation conditions. Write $Z \subseteq G$ for the center of G. Let $A \in C$. We say a deformation M of N to A *has a central character* if Z acts on M via a group homomorphism $Z \to A^\times$. If $Z = \mathbb{Q}_p^\times$ so that $p \in Z$ we say a deformation M of N to A has *uniformizer acting trivially* if p acts trivially on M. We denote these properties by $\mathcal{P}_i, i = 1, 2$ respectively.

Proposition 2.16. *The deformation functor corresponding to $\mathcal{P}_i$ is representable if and only if N is of type $\mathcal{P}_i$.*

Proof: We treat the case $\mathcal{P}_1$ of a central character. The other case is similar. By definition of the G-action $\mathcal{P}_1$ is preserved under base change and it remains to show that property (1) implies (2)-(3) above. Let $A_3 = A_1 \times_{A_0} A_2, M \in D_N(A_3)$ and χ_i the central character of the base change $M_i \in D_N(A_i)$. By topological freeness and compatibility between base change and tensor product we are reduced to $M = A_3$. Recalling that A_3 equals the equalizer in $A_1 \times A_2$ of the maps $A_i \to A_0$ the character $\chi_1 \times \chi_2 : Z \to A_1^\times \times A_2^\times$ is seen to take values in $A_3^\times$. Hence, we obtain (2) and (3) follows by a similar argument. $\square$

3. Proof of the main result

We fix an artinian object $A \in C$ and a compact open subgroup $H \subseteq G$. We identify once and for all $N \simeq k^I$ in $\mathfrak{PM}(k)$ by means of a pseudobasis for N. Invoking the functor (14) we see that the cardinality $|I|$ is an invariant of N. Since base change to k commutes with arbitrary direct products any pseudobasis for $M \in D_\rho(A)$ must have cardinality I, too. This shows

Lemma 3.1. *If $[M, \alpha] \in D_\rho(A)$ then $M \simeq A^I$ in $\mathfrak{PM}(A)$.*

Recall the topological ring $M_I(A) := \mathrm{Hom}_{\mathfrak{PM}(A)}(A^I, A^I)$ (cf. (A.1)). According to Lem. A.2 a (jointly) continuous action

$$A[[H]] \times A^I \longrightarrow A^I$$

is the same as a continuous A-algebra homomorphism $A[[H]] \to M_I(A)$. Having this in mind we follow a strategy of B. Mazur (cf. [Maz89]) to rewrite the functor D_ρ in a more accessible way. Namely, let E_A denote the set of profinite augmented G-representations on the A-module A^I that lift N. It is evidently functorial in A.

Let $\mathrm{GL}_I(A) := M_I(A)^\times$ be the group of units in $M_I(A)$. By our above discussion we may think of an element of E_A as a commutative diagram

(8)
$$\begin{tikzcd} A[[H]] \arrow[r, "cont"] & M_I(A) \\ A[H] \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & A[G] \arrow[u] \end{tikzcd}$$

that reduces via $A \to k$ to the corresponding diagram for N. Note here, that the G-action on any $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is necessarily continuous whence the right vertical arrow. The group $\mathrm{GL}_I(A)$ acts on $M_I(A)$ from the left via conjugation. By acting on the right-upper corner of diagrams (13) this induces an action of the subgroup

$$G_A := \ker\big(\mathrm{GL}_I(A) \to \mathrm{GL}_I(k)\big) = 1 + \prod_I \oplus_I m_A$$

on the set E_A.

Lemma 3.2. *There is a bijection*

$$E_A/G_A \xrightarrow{\cong} D_\rho(A)$$

natural in A.

Proof: Let $\pi : A \to k$ be the residue homomorphism. The map $A^I \mapsto [A^I, 1 \otimes \pi^I]$ induces an injective map from E_A/G_A to $D_\rho(A)$. It is surjective by the above lemma. $\square$

Let us now assume that $N^\vee$ is absolutely irreducible and admissible (note that in this proof and to save notation we reserve the letter ρ for other things than $N^\vee$). Let $\bar\rho$ denote the corresponding element in E_k. Given $\rho \in E_A$ write

$$C(\rho) \subseteq M_I(A)$$

for the A-algebra equal to the centralizer in $M_I(A)$ of the image of $\rho : A[G] \to M_I(A)$. Recall that a surjection $\phi : A \to B$ in C is called a *small extension* if $\ker \phi$ equals a nonzero principal ideal which is annihilated by the maximal ideal of A. Every surjection in C factors as a finite composite of small extensions (cf. [Gou01], Problem 3.1).

Lemma 3.3. *One has $C(\rho) = A$ for all $\rho \in E_A$.*

Proof: We first prove $C(\bar\rho) = k$. Let $f \in C(\bar\rho)$. The Pontryagin Dual $N^\vee$ equipped with the contragredient G-action is an absolutely irreducible admissible smooth G-representation over k (cf. (2)). We fix an open pro-p subgroup H of G. Since $N^\vee$ is admissible the H-invariants $(N^\vee)^H$ form a finite dimensional k-vector space which is nonzero (cf. [Wil98], Lem. 11.1.1) and stabilized by $f^\vee$. After possibly extending scalars any nonzero eigenvector of $f^\vee$ in $(N^\vee)^H$ generates the G-representation $N^\vee$. Hence, $f^\vee$ acts on $N^\vee$ by a scalar and we obtain $C(\bar\rho) = k$. For the general case note that if $A \to B$ is an essential extension in C with kernel generated by $t \in A$ and $\rho \in E_A$ then $\ker\big(M_I(A) \to M_I(k)\big)$ is killed by t (cf. (15)). In this situation the result is an easy consequence of the arguments given in [Gou01], Lem. 3.8. □

As a corollary the functor D_ρ is *continuous* in the usual sense:

Corollary 3.4. *Given $A = \varprojlim_n A_n \in \hat C$ the natural map*

$$D_\rho(A) \xrightarrow{\cong} \varprojlim_n D_\rho(A_n)$$

is a bijection.

Proof: By Lem. 2.5 we may assume that A_n is an artinian quotient of A so that $A_{n+1} \to A_n$ is surjective for all n. It follows that the maps

$$G_{A_{n+1}} \longrightarrow G_{A_n} \tag{9}$$

are surjective for all n. To check surjectivity of the map in question let $([M_n, \alpha_n])_n$ be an element of the projective limit. A straightforward argument, using the surjectivity of (9) shows the existence of isomorphisms $\beta_n : M_{n+1} \otimes_{A_{n+1}} A_n \xrightarrow{\cong} M_n$ compatible with the α_n. Passing to the projective limit using $\varprojlim_n A_n[[H]] = A[[H]]$ (and similarly for $A[H], A[G]$) yields a pseudocompact augmented G-representation on $M := \varprojlim_{\beta_n} M_n$. By Lem. 2.7 M is topologically free over A and therefore the desired preimage. For the injectivity let M, M' be representatives of two classes in $D_\rho(A)$ together with isomorphisms $M_n \simeq M_n'$ for all n which are compatible with reductions. Let ρ, ρ' be the corresponding elements in E_A. A straightforward argument using the surjectivity of

$$C(\rho_{n+1}) \longrightarrow C(\rho_n) \tag{10}$$

(Lem. 3.3) shows that we may assume the isomorphisms $M_n \simeq M_n'$ to be compatible with $A_{n+1} \to A_n$. Passage to the projective limit yields an isomorphism $M \simeq M'$. □

According to Thm. 2.6 and Cor. 3.4 D_ρ is representable if its restriction to C is left exact. Since $D_\rho(k)$ is a singleton this reduces to verify that D_ρ respects fiber

products. Let therefore

$$A_3 = A_1 \times_{A_0} A_2$$

be a fiber product in C. Writing $E_i := E_{A_i}$ and $G_i := G_{A_i}$ and invoking Lem. 3.2 we have to show that the natural map of sets

$$b : E_3/G_3 \to E_1/G_1 \times_{E_0/G_0} E_2/G_2 \tag{11}$$

is a bijection.

Lemma 3.5. *Invoking the fibre product of topological rings the natural maps*

$$M_I(A_3) \xrightarrow{\cong} M_I(A_1) \times_{M_I(A_0)} M_I(A_2) \ , \ A_3[[H]] \xrightarrow{\cong} A_1[[H]] \times_{A_0[[H]]} A_2[[H]] \tag{12}$$

are isomorphisms of topological rings.

Proof: The maps in question are certainly continuous ring homomorphisms. The first map is a topological isomorphism since on the level of topological groups the fibre product $M_I(A_1) \times_{M_I(A_0)} M_I(A_2)$ commutes with the functors $\prod_I$ and $\oplus_I$ in an obvious sense. The second map is a topological isomorphism since on the level of topological rings the fibre product $A_1[[H]] \times_{A_0[[H]]} A_2[[H]]$ commutes with the functor $\varprojlim_{\mathcal{N}}$ in an obvious sense. $\square$

Lemma 3.6. *The map b is surjective.*

Proof: We generalize an argument of M. Dickinson (cf. [Gou01],Appendix 1) to our situation. For this it will be convenient to think of an augmented G-representation ρ in $E_A, A \in C$ as taking values $\rho(g)$ in infinite $I \times I$-matrices. We shall therefore write suggestively $c\rho c^{-1} := c.\rho$ for $c \in G_A$. Let $([\rho], [\sigma]) \in E_1/G_1 \times_{E_0/G_0} E_2/G_2$. Let m_0 be the maximal ideal of A_0. Since A_0 is artinian we have $m_0^n = 0$ for some $n \geq 1$. We first prove by induction on n that $[\rho]$ and $[\sigma]$ have representatives in E_1 and E_2 respectively whose images coincide in E_0.

To do this let $\phi : A_1 \to A_0$ and $\psi : A_2 \to A_0$ be the transition maps in the fiber product. By the induction hypothesis we may assume $n > 1$ and that $\phi_*(\rho) = \psi_*(\sigma)$ mod m_0^{n-1}. Here, we abbreviate $\phi_* = E_A(\phi)$ and similarly for ψ. Pick an element $c \in G_0$ such that $c\phi_*(\rho)c^{-1} = \psi_*(\sigma)$ in E_0. We show that there are elements $g \in G_1, h \in G_2$ such that $\phi_*(g\rho g^{-1}) = \psi_*(h\sigma h^{-1})$. This proves the claim.

Reducing c mod m_0^{n-1} centralizes the image of the reduction $\phi_*(\rho)$ mod m_0^{n-1} and therefore (Lem. 3.3) $c = 1 + l$ with $l \in \prod_I \oplus_I m_0^{n-1}$. Since l has entries in the finite dimensional k-vector space m_0^{n-1} we may apply *mutatis mutandis* [Gou01], App. 1,Lem. 9.3 and arrive at $l = \lambda 1 + \psi(m_2) - \phi(m_1)$ with a scalar $\lambda \in m_0^{n-1}$ and $m_i \in \prod_I \oplus_I A_i$. From now on the claim follows formally from the computations given in loc.cit.

By what we have just shown we may now suppose that $(\rho, \sigma) \in E_1 \times_{E_0} E_2$. The couple gives therefore rise to a continuous homomorphism

$$A_1[[H]] \times_{A_0[[H]]} A_2[[H]] \longrightarrow M_I(A_1) \times_{M_I(A_0)} M_I(A_2)$$

compatible with the G-action. Composing it with the inverses of the maps in (12) yields a preimage in E_3. $\square$

Let again $A_3 = A_1 \times_{A_0} A_2$ in C. For $\rho_i \in E_i$ write

$$G(\rho_i) \subseteq G_i$$

for the stabilizer of ρ_i in G_i. According to Lem. 3.3

$$G(\rho_i) = \big(1 + \prod_I \oplus_I m_{A_i}\big) \cap C(\rho_i) = 1 + m_{A_i}.$$

Lemma 3.7. *The map b is injective.*

Proof: Assume $b([\rho]) = b([\sigma])$ with $[\rho], [\sigma] \in E_3/G_3$. For $\rho \in E_3$ write ρ_i for the image in E_i and similarly for $\sigma \in E_3$. Pick $(g_1, g_2) \in G_1 \times G_2$ with $\rho_i = g_i.\sigma_i$ in E_i. The "top left entry" of the "$I \times I$"-matrix of $g_i \in G_i = 1 + \prod_I \oplus_I m_{A_i}$ lies in $1 + m_{A_i} \subseteq A_i^\times$. Multiplying by a scalar we may therefore assume this entry is equal to one. Let $\bar{g}_i$ denote the image of g_i in G_0. Since $\rho_0 = \sigma_0$ we have $\bar{g}_2^{-1}\bar{g}_1 \in G(\rho_0) = 1 + m_{A_0}$. Comparing top left entries we see that $\bar{g}_1 = \bar{g}_2$ whence $(g_1, g_2) \in G_1 \times_{G_0} G_2 = G_3$. This element conjugates σ to ρ whence $[\rho] = [\sigma]$. $\square$

This completes the proof of main theorem .

4. Functoriality

This section briefly illustrates that the usual functorial properties of the universal deformation ring hold in our setting (cf. [Maz89], 1.3). Let us fix $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ and assume that the associated deformation functor D_ρ is representable.

4.1. **Change of range.** Let I be any set and regard $A \mapsto M_I(A)$ as a functor from $\hat{C}$ to topological $M_I(o)$-algebras. Suppose

$$\delta : M_I \longrightarrow M_J$$

is a morphism of functors. Assume N has a pseudobase indexed by I. Choosing an isomorphism $N \simeq k^I$ we may compose the resulting diagram (cf. (13))

$$\begin{tikzcd} k[[H]] \arrow[r, "cont"] & M_I(k) \\ k[H] \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & k[G] \arrow[u] \end{tikzcd}$$

in the obvious way with $\delta(k)$ and obtain an element $N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ (with a pseudobase indexed by J). Let us assume that the deformation functor $D_{\rho'}$ associated to N' is representable. Let $R = R(\rho)$ and $R' = R'(\rho')$ be the corresponding universal deformation o-algebras. Let $A \in \hat{C}$ with residue homomorphism $\pi : A \to k$. Given $[M, \alpha] \in D_\rho(A)$ we obtain $[M', \alpha'] \in D_{\rho'}(A)$ by applying $\delta(A)$ in the analogous way and using $\delta(\pi) : \delta(A) \to \delta(k)$. This yields a morphism of functors $D_\rho \to D_{\rho'}$ whence a homomorphism of o-algebras

$$r(\delta) : R' \to R$$

which satisfies the homomorphic properties $r(id) = 1,\ r(\delta_2\delta_1) = r(\delta_1)r(\delta_2)$.

Example: If $I = J$ and $\delta = \delta_g$ is given by conjugation with a fixed element $g \in GL_I(o)^\times$

$$\delta_g(A)(.) := g(.)g^{-1}$$

for $A \in \hat{C}$ then, using Lem. 3.3, the homomorphism $r(\delta_g)$ is easily seen to depend only on the representations N and N' (i.e. only on the image $\bar{g} \in GL_I(k)$). We denote it by $r(\rho', \rho)$. It is clearly an isomorphism satisfying the homomorphic properties $r(\rho, \rho) = 1,\ r(\rho', \rho)r(\rho'', \rho') = r(\rho'', \rho)$.

4.2. **Tensor product.** Let $N, N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. Suppose the deformation functors associated to $N,\ N'$ and $N'' := N\,\hat{\otimes}_k N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ are representable. Given $A, A' \in \hat{C}$ with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively the remark at the end of subsection A shows that for any $A, A' \in \hat{C}$ the o-algebra $A'' := A\,\hat{\otimes}_o A'$ lies again in $\hat{C}$ and has maximal ideal $\mathfrak{m}'' = \mathfrak{m}\,\hat{\otimes} A' + A\,\hat{\otimes}\mathfrak{m}'$. Given $[M, \alpha]$ and $[M', \alpha]$ in $D_\rho(A)$ and $D_{\rho'}(A')$ respectively the compatibility of $\hat{\otimes}$ with direct products shows that

$$M'' := M\,\hat{\otimes}_o M' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A'')^{\mathrm{fl}}$$

and

$$M''/\mathfrak{m}''M'' \xrightarrow{\cong} M/\mathfrak{m}M \,\hat{\otimes}_o\, M'/\mathfrak{m}'M' \xrightarrow{\cong} N''$$

where the first isomorphism is canonical and the second induced by $\alpha \otimes \alpha'$. Hence, $[M'', \alpha \otimes \alpha'] \in D_{A''}(N'')$ and we obtain a homomorphism

$$h(\rho, \rho') : R'' \longrightarrow R \,\hat{\otimes}_o R'.$$

We leave it to the reader to check that the system of homomorphisms $(\rho, \rho') \mapsto h(\rho, \rho')$ satisfies the obvious commutativity and associativity relations.

4.3. **Twisting.** Let N, N' and N'' as above. Let M such that $[M, \alpha] \in D_\rho(o)$. We obtain a homomorphism

$$R'' \xrightarrow{h(\rho,\rho')} R \,\hat{\otimes}_o R' \xrightarrow{[M,\alpha]} R'$$

satisfying the formal analogues of the commutativity relations as displayed in [Maz89], p. 9. It formally follows from these relations that if $|I| = 1$ (so that the G-action on N is given by a character) this homomorphism is always an isomorphism in $\hat{C}$ (the *twisting isomorphism*). Thus, the universal deformation ring of N is uniquely determined by the twist-equivalence class of the representation N.

4.4. **Change of group.** Let $\varphi : G \to G'$ be a morphism of locally $\mathbb{Q}_p$-analytic groups. Let $N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ and let N be its pull-back via φ. Clearly, $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. Let us assume that the deformation functors $D_{\rho'}$ and D_ρ are representable. Pulling-back via φ yields a morphism of functors $\varphi_* : D_{\rho'} \to D_\rho$ inducing a homomorphism

$$r(\varphi) : R \longrightarrow R'.$$

The reader may check that the system $\varphi \mapsto r(\varphi)$ is homomorphic in φ and that φ_* is injective if φ is surjective.

4.5. **Change of field.** Let $\iota : o \to o'$ be a morphism of complete local noetherian rings such that the residue extension $\bar{\iota} : k \to k'$ is finite. Let $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ and $N' = \bar{\iota}_G^*(N)$ its base extension to k'. Assume D_N and $D_{\rho'}$ are representable. Tensoring the universal deformation of N with o' over o yields a natural morphism

$$R(\iota) : R' \longrightarrow R \,\hat{\otimes}_o o'$$

of pseudocompact o-algebras homomorphic in ι. It induces an isomorphism on tangent spaces. Indeed, the category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ has enough projectives (being equivalent to smooth G-representations over k, cf. [Vig96], I.5.9) whence a k'-vector space isomorphism $\mathrm{Ext}^1_G(\rho, \rho) \otimes_k k' \xrightarrow{\cong} \mathrm{Ext}^1_G(\rho', \rho')$ by faithful flatness of $\bar{\iota}$.

5. Applications to p-adic Banach space representations

5.1. **Unitary Deformations.** We keep the above notations but assume that o equals the ring of integers in a finite extension K of $\mathbb{Q}_p$. In particular, k is finite of characteristic $p > 0$. Let ϖ be a uniformizer for o. Recall that a *Banach space representation of* G *over* K is a K-Banach space V together with a linear G-action such that the map

$$G \times V \longrightarrow V$$

giving the action is continuous (cf. [ST02]). The representation V is called *unitary* if there is a K-vector space norm $||.||$ on V defining the topology and such that G acts by isometries. In this case the $||.||$-unit ball V^0 is G-stable and

$$\bar{V} := V^0/\varpi V^0$$

defines a smooth G-representation over k. If $\bar{V}$ is admissible smooth we call $(V, ||.||)$ *admissible unitary.* In this case the Pontryagin Dual $N = (\bar{V})^\vee$ lies in $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)$ (cf. (2)) and, hence, our deformation theory applies.

Theorem 5.1. *Let $\rho \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$ be a smooth G-representation over k which is admissible and absolutely irreducible. There is a canonical and natural bijection between the o-valued points of $R(\rho)$ and the set of isomorphism classes of admissible unitary G-Banach space representations V such that $\bar{V} \simeq \rho$.*

Proof: Let $\mathrm{Ban}_G^0(K)^{\leq 1}$ denote the category of unitary G-Banach space representations $(V, ||.||)$ such that $||V|| \subseteq |K|$ equipped with G-equivariant norm decreasing K-linear maps as morphisms. Fix a compact open subgroup $H \subseteq G$. Given $V \in \mathrm{Ban}_H^0(K)^{\leq 1}$ we may equip the o-module $V^d := \mathrm{Hom}_o(V^0, o)$ with the topology of pointwise convergence and the contragredient H-action. The H-equivariant version of the discussion in [ST02], (proof of) Thm. 1.2 shows that $V \mapsto V^d$ establishes an equivalence of categories

$$(.)^d : \mathrm{Ban}_H^0(K)^{\leq 1} \xrightarrow{\cong} \mathfrak{PM}(o[[H]])^{\mathrm{fl}}$$

with admissible objects corresponding to finitely generated $o[[H]]$-modules. On the faithfully embedded subcategory $\mathrm{Ban}_G^0(K)^{\leq 1}$ we deduce an equivalence

$$(.)^d : \mathrm{Ban}_G^0(K)^{\leq 1} \xrightarrow{\cong} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(o)^{\mathrm{fl}}$$

and again, admissible objects correspond to augmented G-representations which are finitely generated over $o[[H]]$. We arrive at a diagram of functors

(13)
$$\begin{tikzcd}
\mathrm{Ban}_G^{0,\mathrm{adm}}(K)^{\leq 1} \arrow[r, "\sim", "(.)^d"'] \arrow[d, "mod\ \varpi"] & \mathrm{Mod}_G^{\mathrm{fg\,aug}}(o)^{\mathrm{fl}} \arrow[d, "mod\ \varpi"] \\
\mathrm{Mod}_G^{\mathrm{adm}}(k) \arrow[r, "\sim", "(.)^\vee"'] & \mathrm{Mod}_G^{\mathrm{fg\,aug}}(k)
\end{tikzcd}$$

where the left perpendicular arrow refers to the functor $V \mapsto \bar{V}$ and the lower horizontal arrows refers to the passage to the Pontryagin dual. A straightforward equivariant version of [Pas], Lem. 5.4 proves the diagram to be commutative. It follows that the functor $(.)^d$ induces a functorial bijection between the set of isomorphism classes of Banach space representations V with $\bar{V} \simeq \rho$ and $D_\rho(o)$. □

5.2. **Principal series representations of $GL_2(\mathbb{Q}_p)$.** We give an application of this result in case $G = GL_2(\mathbb{Q}_p)$. This strongly relies on results of M. Emerton concerning the functor of *ordinary parts* (cf. [Emeb]). To start with let P and $\overline{P}$ be the Borel subgroup of G consisting of upper triangular and lower triangular matrices respectively. Given two smooth characters $\chi_i : \mathbb{Q}_p^\times \longrightarrow A^\times, i = 1, 2$ for $A \in C$ we view $\chi = \chi_1 \otimes \chi_2$ as a smooth character of the diagonal torus T in G in the obvious way. Define

$$\mathrm{Ind}_{\overline{P}}^G(\chi) = \{f : G \to A^\times | f \text{ locally constant}, f(\overline{p}g) = \chi(\overline{p})f(g), p \in \overline{P}, g \in G\}$$

with G-action by right translations. It is a smooth admissible G-representation over A as defined in [Emea]. Finally, let $\epsilon(a) = a|a| \in \mathbb{Z}_p^\times$ for all $a \in \mathbb{Q}_p^\times$ and write $\bar{\epsilon}$ for the induced smooth character $\mathbb{Q}_p^\times \to k^\times$.

If V^0 denotes the unit ball of an element $(V, ||.||)$ in $\mathrm{Ban}_G^{0,\mathrm{adm}}(K)^{\leq 1}$ we write $V_n := V^0/\varpi^n V^0$ and $o_n := o/\varpi^n o$ for all n.

Lemma 5.2. *The o_n-module V_n is faithful for all n.*

Proof: By the same argument as in case $n = 1$ the diagram (13) remains commutative when we replace k by o_n, ϖ by ϖ^n and restrict to topologically free o_n-modules in the lower right corner. But then $(V_n)^\vee$ is topologically free and

therefore V_n is faithful. $\square$

After these preliminaries let us fix smooth characters $\overline{\chi}_i : \mathbb{Q}_p^\times \longrightarrow k^\times, i = 1, 2$ for which $\bar{\chi}_1\bar{\chi}_2^{-1} \neq 1, \bar{\epsilon}$. The G-representation $\mathrm{Ind}_{\overline{P}}^G(\bar{\chi})$ is then admissible and absolutely irreducible (cf. [BL94]). Suppose our chosen Banach space representation V satisfies

$$\bar{V} = V^0/\varpi V^0 \simeq \mathrm{Ind}_{\overline{P}}^G(\bar{\chi}).$$

Lemma 5.3. *Let $n \geq 1$ and suppose there is an isomorphism $V_n \xrightarrow{\cong} \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n)$ with some smooth $o_n^\times$-valued character $\overline{\chi}^n = \chi_1^n \otimes \chi_2^n$. Then there is a smooth $o_{n+1}^\times$-valued character $\overline{\chi}^{n+1} = \chi_1^{n+1} \otimes \chi_2^{n+1}$ and a commutative diagram of smooth G-representations*

$$\begin{tikzcd} V_{n+1} \arrow[r, "\sim", "\varphi_{n+1}"'] \arrow[d, "mod\ \varpi^n"] & \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^{n+1}) \arrow[d, "mod\ \varpi^n"] \\ V_n \arrow[r, "\sim", "\varphi_n"'] & \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n). \end{tikzcd}$$

Proof: This is an adaption of the argument given in [Emeb], Prop. 4.1.5. Multiplication by ϖ^n induces a well-defined surjective morphism of smooth G-representations $V_{n+1}/\varpi V_{n+1} \longrightarrow \varpi^n V_{n+1}$. By Lem. 5.2 it is nonzero and thus injective by irreducibility of the left-hand side. We obtain a short exact sequence of smooth G-representations

$$0 \longrightarrow \mathrm{Ind}_{\overline{P}}^G(\bar{\chi}) \simeq \varpi^n V_{n+1} \longrightarrow V_{n+1} \longrightarrow \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n) \longrightarrow 0.$$

Applying the δ-functor $H^\bullet\mathrm{Ord}_P$ associated to the functor of ordinary parts with respect to P (cf. [Emea], 3.2) yields an exact sequence of smooth T-representations

$$0 \longrightarrow \overline{\chi} \longrightarrow \mathrm{Ord}_P(V_{n+1}) \longrightarrow \overline{\chi}^n \longrightarrow 0$$

using [Emea], Cor. 4.3.5 and [Emeb], Thm. 4.1.3. Applying the exact functor $\mathrm{Ind}_{\overline{P}}^G$ it follows that the unit of the adjunction in [Emea], Thm. 4.4.6 gives an isomorphism $\mathrm{Ind}_{\overline{P}}^G(\mathrm{Ord}_P(V_{n+1})) \xrightarrow{\cong} V_{n+1}$ and that $\mathrm{Ord}_P(V_{n+1})$ must be given by a lift $\overline{\chi}^{n+1}$ of $\overline{\chi}^n$ to $o_{n+1}^\times$. By [Emea], Lem. 4.1.3 the representation $\mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^{n+1})$ is then a lift of $\mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n)$ and, by construction, compatible with the reduction map $V_{n+1} \to V_n$. This proves the claim. $\square$

Given a continuous character $\chi : T \longrightarrow o^\times$ define

$$^{\mathrm{c}}\mathrm{Ind}_{\overline{P}}^G(\chi) = \{f : G \to K^\times | f \text{ continuous}, f(\overline{p}g) = \chi(\overline{p})f(g), p \in \overline{P}, g \in G\}$$

with G acting by right translations. Equipped with a suitable supremum norm it constitutes an admissible unitary G-Banach space representation over K (the *ordinary continuous principal series*, cf. [Sch06], Prop. 2.4).

Corollary 5.4. *If $V \in \mathrm{Ban}_G^0(K)^{\leq 1}$ such that $\bar{V} \simeq \mathrm{Ind}_{\overline{P}}^G(\bar{\chi})$ then V is ordinary principal series with respect to a lift χ of $\overline{\chi}$ to $o^\times$. The isomorphism classes of such V are thus in bijection with the rational points of* Spf $o[[x_1, x_2]]$ *and therefore equal to the set $(\varpi) \times (\varpi)$.*

Proof: Using induction on n together with [Emea], Lem. 4.1.3 implies the first claim. It follows that the set of isomorphism classes of such V is in bijection with $D_{\bar{\chi}}(o)$. Thus the Example following Cor. 2.13 completes the proof. $\square$

Appendix A. Pseudocompact rings

A *pseudocompact ring* is a complete Hausdorff topological unital ring A which admits a system of open neighbourhoods of zero consisting of twosided ideals $\mathfrak{a}$ such that $A/\mathfrak{a}$ has finite length as both a left and a right A-module. In particular, A equals the topological inverse limit of the artinian quotient rings $A/\mathfrak{a}$ each endowed with the discrete topology. A morphism of pseudocompact rings is by definition a continuous unital ring homomorphism. An artinian ring with the discrete topology is evidently pseudocompact. More generally, the topology on a pseudocompact ring A which is (left or right) noetherian is uniquely determined and coincides with the adic topology defined by the Jacobson radical of A.

Let A be a pseudocompact ring. A complete Hausdorff topological left unital A-module M is called *left pseudocompact* if it has a system of open neighbourhoods of zero consisting of submodules M' such that M/M' has finite length as an A-module. A morphism between two pseudocompact left modules is by definition a continuous A-linear map. It necessarily has closed image. Borrowing notation from [SV06] we denote the category of left pseudocompact A-modules by $\mathfrak{PM}(A)$. It is abelian with exact projective limits and the forgetful functor $\mathfrak{PM}(A) \rightarrow \mathfrak{M}(A)$ is faithful and exact and commutes with projective limits (cf. [DG70], IV.3. Thm. 3, [?] Prop. 3.3). An arbitrary direct product of pseudocompact modules is pseudocompact in the product topology. A pseudocompact module M is called *topologically free* if it is topologically isomorphic to a product $\prod_I A$ with some index set I. The set of images in M of the "unit vectors" $(...,0,1,0,...) \in \prod_I A$ under such an isomorphism is called a *pseudobasis* of M.

If A is left noetherian then any finitely generated abstract left A-module has a unique pseudocompact topology. We thus have a natural fully faithful and exact embedding $\mathfrak{M}_{fg}(A) \rightarrow \mathfrak{PM}(A)$.

For the rest of this appendix let us fix a pseudocompact ring A which is commutative. As usual, we then do not distinguish between left and right A-modules. Given a pseudocompact module M over such an A write $M^* := \mathrm{Hom}(M, A)$ for the A-module of morphisms $M \rightarrow A$ in $\mathfrak{PM}(A)$. We obtain a functor

$$M \mapsto M^* \tag{14}$$

between $\mathfrak{PM}(A)$ and $\mathfrak{M}(A)$ that changes direct products into direct sums. If A is artinian this establishes an anti-equivalence of categories between projective objects in $\mathfrak{PM}(A)$ and $\mathfrak{M}(A)$ respectively (cf. [DG70], 0.2.2). In general, if $A^I := \prod_I A$ denotes a topologically free module on a pseudobasis I and

$$M_I(A) := \mathrm{Hom}_{\mathfrak{PM}(A)}(A^I, A^I)$$

its endomorphism ring we obtain an isomorphism of abstract A-modules

$$M_I(A) \xrightarrow{\cong} \prod_I (A^I)^* = \prod_I \oplus_I A, \tag{15}$$

natural in A. By transport of structure we equip $M_I(A)$ with the product topology induced by the discrete topology on each factor $\oplus_I A$.

Lemma A.1. *The correspondance $A \mapsto M_I(A)$ is a functor from artinian pseudocompact rings to topological rings.*

Proof: Using (15) we may regard elements in the ring $M_I(A)$ as infinite $I \times I$-matrices. On coefficients the ring operations are then expressed by the usual formulas whence $M_I(A)$ is a topological ring. Functoriality is clear. $\square$

If M, N denote two pseudocompact A-modules define the A-module

$$M \hat{\otimes}_A N := \varprojlim_{M',N'} M/M' \otimes_A N/N' \tag{16}$$

where M' and N' run through the open submodules of M and N respectively. If each $M/M' \otimes_A N/N'$ is endowed with the discrete topology the projective limit topology makes $M \hat{\otimes}_A N$ a pseudocompact o-module. Indeed, given M', N' as above there exists an open ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a}M \subseteq M'$ and $\mathfrak{a}N \subseteq N'$ so that $M/M' \otimes_A N/N'$ is a finitely generated module over the artinian ring $A/\mathfrak{a}$ and therefore of finite A-length. The binary operation $\hat{\otimes}_A$ on $\mathfrak{PM}(A)$ is associative and commutative with A as a unit object and functorial in both variables. It commutes with projective limits and direct products (cf. [DG70], VIIb. 0.3.5/6). Now let $\phi : A \to B$ be a morphism between commutative pseudocompact rings and $M \in \mathfrak{PM}(A)$. Define the B-module $M \hat{\otimes}_A B$ by the exact analogue of formula (16). Arguing as above shows $M \hat{\otimes}_A B$ to be a pseudocompact B-module. We obtain a "base change" functor

$$\phi^* : \mathfrak{PM}(A) \longrightarrow \mathfrak{PM}(B)$$

which commutes with tensor products, projective limits and direct products (cf. [DG70], 0.5).

A *pseudocompact algebra over A* is a topological unital A-algebra which admits a system of open neighbourhoods of zero consisting of twosided ideals $\mathfrak{b}$ such that $B/\mathfrak{b}$ has finite length as an A-module. Such a ring B is evidently pseudocompact and a B-module has finite length if and only if it has finite length as an A-module. Hence we have natural faithful and exact forgetful functor

$$\phi_* : \mathfrak{PM}(B) \longrightarrow \mathfrak{PM}(A) \tag{17}$$

letting $\phi : A \to B$ denote the structure map. An A-algebra which is of finite length as A-module is evidently a pseudocompact A-algebra in the discrete topology.

Lemma A.2. *Let A be artinian, B be a pseudocompact A-algebra and $A^I \in \mathfrak{PM}(A)$ topologically free. Assume there is a homomorphism of abstract A-algebras $\tilde{f} : B \to M_I(A)$. The induced map $f : B \times A^I \to A^I$ is continuous if and only if $\tilde{f}$ is continuous.*

Proof: By definition $\tilde{f}(b) = f(b, .)$ for all $b \in B$. We denote by π_i for all i the projection of A^I and $M_I(A) = \prod_I (A^I)^*$ onto the i-th factor A and $(A^I)^*$ respectively. Put $f_i = \pi_i \circ f$ and $\tilde{f}_i = \pi_i \circ \tilde{f}$. Now suppose that $\tilde{f}$ is continuous. It suffices to show that f_i is continuous at $(0,0)$ for all i. Fix i. The map $\tilde{f}_i = \pi_i \circ f(0, .) : A^I \to A$ is continuous with discrete target whence $\pi_i \circ f(0, W) = 0$ for some open neighbourhood $0 \in W \subseteq A^I$. It follows that $\tilde{f}_i(0) \in B(W) := \{h \in (A^I)^* : h(W) = 0\}$ and therefore $V := \tilde{f}_i^{-1}(B(W)) \subseteq B$ is an open neighbourhood of 0. Thus, $0 = \tilde{f}_i(V)(W) = f_i(V, W)$ and f_i is continuous at $(0,0)$.

Conversely, suppose that f is continuous. It suffices to show that $\tilde{f}_i$ is continuous at $0 \in B$ for all i. Fix i. Since f_i is continuous with discrete target we may choose an open neighbourhood $0 \in V' \subseteq B$ and an open A-submodule $W \subseteq A^I$ of finite colength such that $\tilde{f}_i(V')(W) = f_i(V', W) = 0$. Let $e_1, ..., e_m$ be A-module generators for A^I/W. Again by continuity of f_i and since f defines a B-action on A^I there is an open neighbourhood $0 \in V_j \subseteq V'$ such that $f_i(V_j, e_j) = 0$ for $j = 1, ..., m$. Putting $V = \cap V_j$ yields $\tilde{f}_i(V) = 0$ whence $\tilde{f}_i$ is continuous at 0. □

Remark: If the pseudocompact topology on A is compact the topology on $M_I(A)$ is easily seen to coincide with the compact-open topology and the preceding result follows from [Bou89], Thm. X.§3.3.

A morphism between two pseudocompact algebras over A is by definition a continuous unital A-algebra homomorphism. The category of pseudocompact A-algebras has projective limits and finite inductive limits. In particular, if $B \to C$

and $B \to D$ are morphisms between pseudocompact A-algebras the amalgamated sum of C and D over B has $C \hat{\otimes}_B D$ as underlying pseudocompact A-module together with the obvious multiplication (cf. [DG70], VII.b 0.4.1). Finally, if B is a pseudocompact A-algebra and C is a pseudocompact B-algebra then evidently C is a pseudocompact A-algebra.

References

[BL94] L. Barthel and R. Livné, *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J. **75** (1994), no. 2, 261–292.

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Math., vol. 126, Springer-Verlag, New York, 1991.

[Bos91] Nigel Boston, *Explicit deformation of Galois representations*, Invent. Math. **103** (1991), no. 1, 181–196.

[Bou89] Nicolas Bourbaki, *General topology. Chapters 5–10*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989, Translated from the French, Reprint of the 1966 edition.

[Bru66] Armand Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Algebra **4** (1966), 442–470.

[DdSMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.

[DG70] M. Demazure and A. Grothendieck, *Schémas en groupes i*, Lecture Notes in Math., vol. 151, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[Emea] M. Emerton, *Ordinary parts of admissible representations of p-adic reductive groups i. definitions and first properties*, Preprint 2009. To appear in: Astérisque.

[Emeb] ———, *Ordinary parts of admissible representations of p-adic reductive groups ii. derived functors*, Preprint 2007. To appear in: Astérisque.

[Gou01] Fernando Q. Gouvêa, *Deformations of Galois representations*, Arithmetic algebraic geometry (Park City, UT, 1999), IAS/Park City Math. Ser., vol. 9, Amer. Math. Soc., Providence, RI, 2001, Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton, pp. 233–406.

[Gro] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. II. Le théorème d'existence en théorie formelle des modules*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, pp. Exp. No. 195, 369–390.

[Maz89] Barry Mazur, *Deforming Galois representations*, Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 385–437.

[Maz97] ———, *An introduction to the deformation theory of Galois representations*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 243–311.

[Nag62] Masayoshi Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[Pas] V. Paskunas, *Admissible unitary completions of locally* $\mathbb{Q}_p$*-rational representations of* $\mathrm{GL}_2(F)$, Preprint 2008. Available at: http://arxiv.org/pdf/0805.1006v1.

[RZ00] Luis Ribes and Pavel Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 40, Springer-Verlag, Berlin, 2000.

[Sch68] Michael Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[Sch06] P. Schneider, *Continuous representation theory of p-adic Lie groups*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1261–1282.

[ST02] P. Schneider and J. Teitelbaum, *Banach space representations and Iwasawa theory*, Israel J. Math. **127** (2002), 359–380.

[SV06] Peter Schneider and Otmar Venjakob, *On the codimension of modules over skew power series rings with applications to Iwasawa algebras*, J. Pure Appl. Algebra **204** (2006), no. 2, 349–367.

[Vig96] Marie-France Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec* $l \neq p$, Progress in Mathematics, vol. 137, Birkhäuser Boston Inc., Boston, MA, 1996.

[Wil98] John S. Wilson, *Profinite groups*, London Mathematical Society Monographs. New Series, vol. 19, The Clarendon Press Oxford University Press, New York, 1998.

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

E-mail address: toschmid@math.uni-muenster.de